

D-MODULE GENERATION IN POSITIVE CHARACTERISTIC VIA FROBENIUS DESCENT

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ABSTRACT. In this note I show that for most F -finite regular rings R of positive characteristic the localization R_f at an element $f \in R$ is generated by f^{-1} as a D_R -module. This generalizes and gives an alternative proof of the results in [1] where the result is proven for the polynomial ring, thereby answering questions raised in [1] affirmatively. The proof given here is a surprisingly simple application of Frobenius descent, a brief but thorough discussion of which is also included. Furthermore I show how essentially the same technique yields a quite general criterion for obtaining D_R -module generators of a unit $R[F]$ -module.

1. R_f IS D_R -GENERATED BY f^{-1}

Throughout this paper R will denote a noetherian regular ring. This note was created after hearing about the result of Alvarez Montaner and Lyubeznik in [1]. For the case of $R = k[x_1, \dots, x_n]$ they show the following theorem:

Theorem 1.1. *Let R be a regular F -finite ring of positive characteristic, which is essentially of finite type over a regular local ring. Let $f \in R$ be a nonzero element. Then the D_R -module R_f is generated by f^{-1} .*

What is surprising about this result is that in characteristic zero it is incorrect. There, the D_R -generation of R_f is governed by the Bernstein-Sato polynomial of f , and therefore reflects the geometry of the hypersurface defined by $f = 0$. Thus this is another instance where D_R -modules appear coarser in positive characteristic than in characteristic zero, cf. for example [4, 11].

The proof of Theorem 1.1 given here relies on two results which are central to the study of differential operators in positive characteristic. The first one is the fact that the localization R_f has finite length as a D_R -module, which is implied by the following more general result of Lyubeznik.

Theorem 1.2 ([12, Theorem 5.7]). *Let R be regular, F -finite and essentially of finite type over a F -finite ring. Let M be a finitely generated unit $R[F]$ -module. Then M has finite length as a D_R -module.*

A unit $R[F]$ -module¹ is an R -module M together with an isomorphism $\vartheta_M : F^*M \rightarrow M$. Note that $F : \operatorname{Spec} R \rightarrow \operatorname{Spec} R$ is the absolute Frobenius map, which is the identity on the underlying topological space and the p th power map on the structure sheaf. In particular on global sections this is

¹What we call here a *finitely generated unit $R[F]$ -module* is called an *F -finite R -module* in [12]

just the Frobenius map $F : R \rightarrow R$ raising r to r^p , which, by abuse, is also denoted by the letter F . The assumption that R be F -finite just means that the Frobenius is a finite map.

The functors F^* and F_* are just pullback and pushforward along the Frobenius. Concretely, one can think of $F^*M = R_F \otimes M$ where the tensor product on the left is via the Frobenius $F : R \rightarrow R$. Similarly F_*M is just M as an abelian group with R structure twisted by the Frobenius.

By adjointness, a R -linear map $\vartheta : F^*M \rightarrow M$ is equivalent to an R -linear map $F_M : M \rightarrow F_*M$ which in turn is nothing but a structure of a module over the noncommutative ring

$$R[F] \stackrel{\text{def}}{=} \frac{R\langle F \rangle}{\{r^p F - Fr \mid r \in R\}}$$

on M . One calls (M, ϑ) *finitely generated* if it is finitely generated as a module over the ring $R[F]$. For $f \in R$ the localization R_f is a finitely generated unit $R[F]$ -module.²

It was shown in [12] that a (finitely generated) unit $R[F]$ -module (M, ϑ) carries a natural structure of a D_R -module. Furthermore, the structural map $\vartheta : F^*M \rightarrow M$ is then D_R -linear, where the D_R -structure on F^*M is due to the Theorem 1.3 below.

The second crucial ingredient is the so called *Frobenius descent*. For the convenience of the reader I will include a (very) brief treatment of this powerful and widely applicable technique at the end of this paper. Most relevant here is the following consequence:

Theorem 1.3. *Let R be a regular and F -finite ring of positive characteristic. Then the Frobenius functor is an autoequivalence of the category of D_R -modules. In particular, for a D_R -module M the module F^*M carries a natural D_R -module structure.*

Proof of Theorem 1.1. For any D_R -submodule $M \subseteq R_f$ one identifies the D_R -module F^*M with its isomorphic image in R_f via the natural D_R -module isomorphism $\vartheta : F^*R_f \rightarrow R_f$. Then F^*M is the D_R -submodule of R_f consisting of the elements rm^p for $r \in R$ and $m \in M \subseteq R_f$.

Let $M = D_R f^{-1}$ and let me point out that $M \subseteq F^*M$: Because F^*M is a D_R -submodule of R_f (by Frobenius descent) it is enough to show that $f^{-1} \in F^*M$. Since F^*M (as a submodule of R_f) consists precisely of the elements rm^p for $r \in R$ and $m \in M$ we may use $r = f^{p-1}$ and $m = f^{-1}$ to conclude that $f^{-1} = rm \in F^*M$.

Now, by repeated application of the Frobenius we get an increasing chain of D_R -submodules of R_f :

$$(**) \quad M \subseteq F^*M \subseteq F^{2*}M \subseteq F^{3*}M \subseteq \dots$$

Since $f^{-1} \in M$ it follows that $f^{-p^e} = F^e(f^{-1})$ is an element of $F^{*e}M$ which shows that the union of the chain must be all of R_f .

²With the identification $F^*R_f = R_F \otimes R_f$ the map $R_f \rightarrow F^*R_f$ sending $\frac{a}{b}$ to $ab^{p-1} \otimes \frac{1}{b}$ is inverse to the natural map $\vartheta : F^*R_f \rightarrow R_f$. The corresponding Frobenius action F is just raising to the p th power. Therefore R_f is generated as an $R[F]$ -module by f^{-1} . Thus R_f is a finitely generated unit $R[F]$ -module.

Thus it is enough to show that $M = F^*M$ since then the limit system is constant and $M = R_f$ as claimed. Let us suppose otherwise, that is assume that the inclusion $M \subsetneq F^*M$ is strict. By Frobenius descent, *all* the inclusions of $(**)$ must be strict. But this contradicts the finite length of R_f as a D_R -module. \square

Remark 1.4. With this result I am able to answer the two (related) questions raised at the end of [1]. Firstly, they asked whether their result for the polynomial ring would also hold for the power series ring; this case is covered by the above theorem.

Secondly, in their proof is only one step ([1, Lemma 3.5]) which does not hold in the complete case. In fact one easily sees that the truth of their Lemma 3.5 for a power series ring is in fact equivalent to Theorem 1.1 for a power series ring. Since the latter was established above, Lemma 3.5 in [1] is therefore also valid in the complete case.

2. D_R -GENERATORS OF UNIT $R[F]$ -MODULES

The above result also follows from a more general observation.

Theorem 2.1. *Let R be a regular F -finite ring of positive characteristic, which is essentially of finite type over a regular local ring. Let N be a finitely generated unit $R[F]$ -module. Suppose $M \subseteq N$ is a D_R -submodule such that $M \subseteq F^*M$. Then M is a unit $R[F]$ -submodule.*

Proof. Once more I identify $F^*M \subseteq F^*N$ with its isomorphic image in N via the structural isomorphism $\vartheta : F^*N \rightarrow N$ of the unit $R[F]$ -module N . Then, M being a unit $R[F]$ -submodule just means that the inclusion $M \subseteq F^*M$ is in fact an equality. Assuming otherwise we apply Frobenius to the strict inclusion $M \subsetneq F^*M$. By Frobenius descent we conclude that all the inclusions $F^{e*}M \subsetneq F^{(e+1)*}M$ are strict as well. The resulting strictly increasing infinite chain

$$M \subsetneq F^*M \subsetneq F^{2*}M \subsetneq F^{3*}M \subsetneq \dots$$

contradicts the finite length of N as a D_R -module. \square

This result was inspired by a result in [6], Proposition 15.3.4, which (in the notation of Theorem 2.1) states that if $F^*M \subseteq M$ then M is also a unit $R[F]$ -submodule.

To obtain Theorem 1.1 from this just note that $M = D_R f^{-1}$ satisfies $M \subseteq F^*M$ and contains the $R[F]$ -module generator f^{-1} of R_f .

Corollary 2.2. *With the same assumptions as in Theorem 2.1, if n_1, \dots, n_t are generators of a root³ of the finitely generated unit $R[F]$ -module N , then n_1, \dots, n_t generate N as a D_R -module.*

Proof. By Theorem 2.1 it is enough to check that the D_R -submodule $M \stackrel{\text{def}}{=} D_R \langle n_1, \dots, n_t \rangle$ satisfies $M \subseteq F^*M$ and contains the $R[F]$ -module generators n_1, \dots, n_t of N . The second statement is trivial and for the first one

³An R -submodule N_0 of a unit $R[F]$ -module N is called a *root*, if N_0 is finitely generated as an R -module, $N_0 \subseteq F^*N_0$ and $\bigcup_e F^{e*}N_0 = N$. The existence of a root is equivalent to N being finitely generated as a unit $R[F]$ -module.

observes that, by definition of root, one can write $n_i = \sum r_j F_N(n_j)$ for some $r_j \in R$. Noting that $F_N(n_j) \in F^*M$ we conclude $n_i \in F^*M$ for all i as required. \square

3. FROBENIUS DESCENT

Frobenius descent (in the basic form used here) is based on the simple fact that a ring R is Morita equivalent to the matrix algebra of $n \times n$ matrices with entries in R . That is R and $\text{Mat}_{n \times n}(R)$ have equivalent module categories.

The application of this basic observation to the study of D_R -modules in positive characteristic is very successful as the works of S.P. Smith [14, 13], B. Haastert [7, 8] and R. Bøgvad [5] demonstrate.⁴ The ultimate generalization is the treatment of Berthelot [2].

The way this Morita equivalence enters the picture comes from the fact that in characteristic $p > 0$, the ring of differential operators of an F -finite ring R is the union

$$D_R = \bigcup_e D_R^{(e)}$$

where $D_R^{(e)} = \text{End}_{R^{p^e}}(R)$ consist of endomorphisms $\varphi \in \text{End}_{\mathbb{Z}}(R)$ which are linear over the subring R^{p^e} of p^e th powers of R (see for example [15] or [3, Chapter 3.1]). Replacing the R^{p^e} linear inclusion $R^{p^e} \subseteq R$ with the R -linear $F : R \rightarrow F_*^e R$ we identify $D_R^{(e)}$ with $\text{End}_R(F_*^e R)$. If in addition R is regular, a basic result of Kunz [10] implies that $F_*^e R$ is a locally free R -module of finite rank, thus locally, $D_R^{(e)}$ is indeed just a matrix algebra over R .

The aim of this section⁵ is to make the resulting Morita equivalence between R and $D_R^{(e)} = \text{End}_R(F_*^e R)$ explicit. Concretely I want to show that it is induced by the Frobenius. This is the content of the following proposition.

Proposition 3.1 (Frobenius Descent). *Let R be regular and F -finite. Then F^{e*} is an equivalence of categories between the category of R -modules and the category of $D_R^{(e)}$ -modules. Its inverse functor is given by*

$$T^e(_) \stackrel{\text{def}}{=} \text{Hom}_R(F_*^e R, R) \otimes_{D_R^{(e)}} _.$$

Proof. Let us view $\text{Hom}_R(F_*^e R, R)$ as an R - $D_R^{(e)}$ -bimodule. The action is by post- and pre-composition respectively. That is for $\delta \in D^{(e)}$, $\varphi \in \text{Hom}_R(F_*^e R, R)$ and $r \in R$ the product $r \cdot \varphi \cdot \delta$ is given by the composition

$$F_*^e R \xrightarrow{\delta} F_*^e R \xrightarrow{\varphi} R \xrightarrow{r} R.$$

⁴A predecessor of it is the so-called Cartier descent as described, for example, in Katz [9, Theorem 5.1]. It states that F^* is an equivalence between the category of R -modules and the category of modules with integrable connection and p -curvature zero. The inverse functor of F^* on a module with connection (M, ∇) is in this case given by taking the horizontal sections $\ker \nabla$ of M . As an R -module with integrable connection and p -curvature zero is nothing but a $D_R^{(1)}$ -module, Cartier descent is just the case $e = 1$ of Proposition 3.1.

⁵This section is an excerpt of Chapter 3.2 of [3]. We state and proof the basic result but for all the straightforward (but tedious) compatibilities one has to check we refer to [3]. Obviously, in this section no originality beyond the exposition is claimed.

Also $F_*^e R$ itself is viewed as a $D_R^{(e)}\text{-}R$ -bimodule. The left $D_R^{(e)}$ -structure is clear by definition of $D_R^{(e)} = \text{End}_R(F_*^e R)$ and the the right R -structure is via Frobenius $F : R \rightarrow F_*^e R$. Thus, the Frobenius functor $F^{e*}(_)$ can be identified with $F_*^e R \otimes_R _$ and thus $F^* M$ naturally carries the structure of a $D_R^{(e)}$ -module. In this manner it is clear that the described associations are functors between the claimed categories as they are just tensoring with an appropriate bimodule.

It remains to show that they are canonically inverse to each other. For this observe that the natural map

$$\Phi : F_*^e R \otimes_R \text{Hom}_R(F_*^e R, R) \rightarrow D_R^{(e)}$$

given by sending $a \otimes \varphi$ to the composition

$$F_*^e R \xrightarrow{\varphi} R \xrightarrow{F^e} F_*^e R \xrightarrow{a} F_*^e R$$

is an isomorphism (of bi-modules) by the fact that $F_*^e R$ is a locally free and finitely generated R -module (Hom commutes with finite direct sums in the second argument). Thus Φ is a natural transformation of $F_*^e R \otimes_R \text{Hom}_R(F_*^e R, R) \otimes_{D_R^{(e)}} _$ to the identity functor on $D_R^{(e)}\text{-mod}$.

Conversely it is equally easy to see that the map

$$\Psi : \text{Hom}_R(F_*^e R, R) \otimes_{D_R^{(e)}} F_*^e R \rightarrow R$$

given by sending $\varphi \otimes a$ to $\varphi(a)$ is also an isomorphism: After a local splitting π of $F^e : R \rightarrow F_*^e R$ is chosen (it exists by local freeness of $F_*^e R$ over R), its inverse is given by $a \mapsto \pi \otimes F^e(a)$. \square

Remark 3.2. It is possible to give a more explicit description of T^e as follows. Let J_e be the left ideal of $D_R^{(e)}$ consisting of all operators δ such that $\delta(1) = 0$. Then

$$T^e(M) \cong \text{Ann}_M J_e.$$

If one has a splitting π_e of the Frobenius $F^e : R \rightarrow F_*^e R$ there is yet another description of T^e . Note that $(F^e \circ \pi_e)$ is a map

$$F_*^e R \xrightarrow{\pi_e} R \xrightarrow{F^e} F_*^e R$$

such that it can be viewed as a differential operator in $D_R^{(e)}$ and therefore acts on any $D_R^{(e)}$ -module M . One can show that

$$T^e(M) \cong (F^e \circ \pi_e)(M)$$

and that $(F^e \circ \pi_e)(M) \subseteq M$ is independent of the chosen splitting π_e . These statements are also verified in [3, Chapter 2.3].

Proposition 3.1 implies that the categories of $D_R^{(e)}$ -modules for all e are equivalent since each single one of them is equivalent to $R\text{-mod}$. The functor giving the equivalence between $D^{(f)}\text{-mod}$ and $D^{(f+e)}\text{-mod}$ is, of course, F^{e*} . Concretely, to understand the $D^{(f+e)}$ -module structure on $F^{e*} M$ for some $D^{(f)}$ -module M , we write $M \cong F^{f*} N$ for $N = T^f(M)$. Then $F^{e*} M = F^{(f+e)*} N = R^{(f+e)} \otimes N$ carries obviously a $D^{(f+e)}$ -module structure with $\delta \in D^{(f+e)}$ acting via $\delta \otimes \text{id}_N$.

Since the union $\bigcup D_R^{(e)}$ is just the ring of differential operators D_R of R this shows (after the obvious compatibilities are checked, which is straightforward and carried out in [3, Chapter 3.2]) that F^{e*} is in fact an auto-equivalence of the category of D_R -modules:

Proposition 3.3. *Let R be regular and F -finite. Then F^{e*} is an equivalence of the category of D_R -modules with itself.*

Finally we specialize to the case that M is a unit $R[F]$ -module. Therefore it naturally carries a D_R -module structure [12], and so does F^*M , via Frobenius descent. The following lemma shows that these structures are compatible:

Lemma 3.4. *Let R be regular and F -finite and (M, ϑ) be a unit $R[F]$ -module. Then $\vartheta : F^*M \rightarrow M$ is a map of D_R -modules.*

Proof. Again I omit the straightforward verification of this and instead refer to [3, Chapter 3.2] \square

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